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ω^* HAS (ALMOST) NO CONTINUOUS IMAGES

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ABSTRACT

We prove that the following statement follows from the Open Colouring Axiom (OCA): if X is locally compact σ-compact but not compact and if its Čech–Stone remainder X^* is a continuous image of ω^*, then X is the union of ω and a compact set. It follows that the remainders of familiar spaces like the real line or the sum of countably many Cantor sets need not be continuous images of ω^*.

Introduction

In [6] Parovičenko proved two results that have received the status of classical in the study of compactifications. The first states that every compact space of weight $\aleph_1$ is a continuous image of the space ω^* — the Čech–Stone remainder of ω — and consequently that the Continuum Hypothesis (CH) implies that every compact space of weight $\mathfrak{c}$ is a continuous image of ω^*. The second result states that CH implies that ω^* is, up to homeomorphism, the only compact zero-dimensional space without isolated points that is an F-space (disjoint open F_σ-subsets have disjoint closures) in which nonempty G_δ-subsets have nonempty interiors — a space with these properties is now generally known as a **Parovičenko space**.

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It is largely because of Parovičenko's second theorem that the space ω^* is very well understood under CH — see, for example, van Mill's survey [5]. One of the reasons for this success is that very many Čech–Stone remainders are Parovičenko spaces: if X is compact and zero-dimensional and of weight $\mathfrak{c}$ or less, then $(\omega \times X)^*$ is a Parovičenko space.

There are examples to show Parovičenko's results cannot be improved upon: (1) if one adds $\aleph_2$ (or more) Cohen reals to a model of CH then the ordinal space $\omega_2 + 1$ is not a continuous image of ω^* (this follows from results of Kunen in [4]); (2) there are two Parovičenko spaces one of which has a point of character $\aleph_1$ whereas in the other every point has character $\mathfrak{c}$, thus showing that Parovičenko's second theorem is equivalent to CH (van Douwen and van Mill [1]); and (3) if every homeomorphism of ω^* is trivial then ω^* and $\big(\omega \times (\omega + 1)\big)^*$ are not homeomorphic (see van Mill [5]; the antecedent was proved consistent by Shelah in [7]).

In this paper we show it consistent that ω^* is the only 'naturally occurring' Parovičenko space that is a continuous image of ω^*. By naturally occurring we mean: of the form X^*, where X is locally compact, σ-compact but not compact.

To be precise our main theorem reads as follows.

Main Theorem (OCA): *If X is locally compact, σ-compact but not compact and if X^* is a continuous image of ω^*, then X is the disjoint sum of ω and a compact space. In short, X^** **is** *ω^*.*

That X must be locally compact and not compact is clear: X^* must be compact and nonempty. The assumption of σ-compactness is there to guarantee the Parovičenko properties (except possibly for zero-dimensionality); this is due to Fine and Gillman [3]. We also use the σ-compactness of X in our proof: at one point we need a perfect map from X into the real line.

The theorem is false without some extra assumption on X; the ordinal space ω_1, for example, is locally compact, not compact and its one-point remainder is clearly a continuous image of ω^* but ω_1 is definitely not the disjoint sum of ω and a compact space. In fact, every compact space K can be realized as the remainder of some pseudocompact space, namely $\omega_1 \times K$. We should like to have a version of our main theorem for nonpseudocompact spaces in general but at present we do not know what such a version should say.

The OCA in the statement of the theorem is the **Open Colouring Axiom** from Todorčević [8]. It reads as follows: if X is separable and metrizable and if $[X]^2 = K_0 \cup K_1$, where K_0 is open in the product topology of $[X]^2$, then **either** X has an uncountable K_0-homogeneous subset Y **or** X is the union of a countably many K_1-homogeneous subsets.

One can deduce OCA from the **Proper Forcing Axiom** (PFA) or prove it consistent in an ω_2-length countable support proper iterated forcing construction, using $\diamond$ on ω_2 to predict all possible subsets of the Hilbert cube and all possible open colourings of these.

We have organized the paper as follows.

In Section 1 we reduce the problem to showing that the particular remainder $\mathbb{D}^*$ is not a continuous image of ω^*, where $\mathbb{D}$ denotes the space $\omega \times (\omega + 1)$. At one point in this reduction we shall require a known consequence of OCA. This suggests an obvious question, which we shall come back to at the end of the paper.

In Section 2 we show that a continuous surjection from ω^* onto $\mathbb{D}^*$ cannot be simple, where 'simple' means that it is induced by a map from $\mathbb{D}$ to $[\omega]^{<\omega}$.

Finally then, in Section 3 we show that, under OCA, all surjections from ω^* onto $\mathbb{D}^*$ must be simple in the sense above. This proof largely parallels Veličković's proof, from [9] and [10], that under OCA +MA all autohomeomorphisms of ω^* are trivial — we shall indicate the main differences, how to avoid the use of MA for example.

1. A reduction

In this section we reduce our problem to one particular remainder. We shall, for the nonce, call a space ω-**like** if it is the disjoint sum of ω and a compact space.

We shall show that it suffices to prove that $\mathbb{D}^*$ is not a continuous image of ω^*, where, as agreed above, $\mathbb{D} = \omega \times (\omega + 1)$. We do this in two steps.

First we show, by elementary topological means, that if X is not ω-like then X^* maps onto $\mathbb{D}^*$ or $\mathbb{H}^*$, where $\mathbb{H}$ is the half line $[0, \infty)$.

After this we show **assuming** OCA that if ω^* maps onto $\mathbb{H}^*$ then it also maps onto $\mathbb{D}^*$ — we do not know whether this implication holds in ZFC, see Section 4. Thus, assuming OCA, if ω^* maps onto X^* for some non ω-like space X then it maps onto $\mathbb{D}^*$.

1.1 First reduction. Let X be a locally compact σ-compact non-compact space that is not ω-like. Write X as an increasing union of compact subsets: $X = \bigcup_{n\in\omega} X_n$, with $X_n \subseteq \operatorname{int} X_{n+1}$ for all n.

Because X is not ω-like no complement $X \setminus X_n$ is discrete; we can therefore, upon taking a subsequence of the X_n, assume that no difference $X_{n+1} \setminus X_n$ is discrete. Choose for each n a non-isolated point x_n of $X_{n+1} \setminus X_n$ and a neighbourhood U_n of x_n whose closure is contained in $X_{n+1} \setminus X_n$.

It is now an easy matter to find a continuous function $\tilde{f}$ from $\bigcup_{n\in\omega}\operatorname{cl} U_n$ to $\mathbb{H}$ such that $\tilde{f}(x_n)=n+1$, $\tilde{f}[\operatorname{cl} U_n]\subseteq[n,n+1]$ and $\tilde{f}[\operatorname{cl} U_n]$ contains a sequence S_n in $[n,n+1)$ that converges to $n+1$. Extend $\tilde{f}$ to a continuous function $f: X\to\mathbb{H}$ such that $f[X_{n+1}\setminus X_n]\subseteq[n,n+1]$ for all n. Note that $f[X]$ is closed in $\mathbb{H}$ and that f is a perfect map.

Now there are two cases to consider:

1. We can find a retraction of $f[X]$ onto the union of ω and subsequences of the S_n; in this case we obtain a perfect map from X onto $\mathbb{D}$ and hence from X^* onto $\mathbb{D}^*$.
2. There is no such retraction; in this case there are infinitely many n such that $f[X]$ contains an interval around $n+1=f(x_n)$. We can then map $f[X]$ onto $\mathbb{H}$ by a perfect map and hence we can map X^* onto $\mathbb{H}^*$.

1.2 SECOND REDUCTION. We assume we are in the second case mentioned above; so we merely know that ω^* maps onto $\mathbb{H}^*$. Because $\mathbb{H}^*$ is connected and $\mathbb{D}^*$ is not we cannot conclude automatically that ω^* maps onto $\mathbb{D}^*$. We show that ω^* maps onto $\mathbb{D}^*$ anyway in two steps.

STEP 1: Let $f\colon\omega^*\to\mathbb{H}^*$ be a continuous surjection and fix a clopen set C in ω^* such that $f^{\leftarrow}\bigl[(\bigcup_n[4n,4n+1])^*\bigr]\subseteq C\subsetneq f^{\leftarrow}\bigl[(\bigcup_n[4n-1,4n+2])^*\bigr]$. It is now clear that we can find a retraction of $f[C]$ onto $\bigl(\bigcup_n[4n,4n+1]\bigr)^*$ and hence that we can map ω^* onto $\mathbb{M}^*$, where $\mathbb{M}=\omega\times[0,1]$.

STEP 2: We now show how to obtain a map from ω^* onto $\mathbb{D}^*$, given a map h from ω^* onto $\mathbb{M}^*$. As mentioned above we shall need the Open Colouring Axiom to accomplish this.

The idea will be to find a clopen set C in ω^* that is mapped onto F^* by h, where $F=\omega\times F'$ and $F'=\{0\}\cup\bigcup_i[2^{-2i-1},2^{-2i}]$. This will suffice because it is easily seen that F^* maps onto $\mathbb{D}^*$.

To this end let $G'=\{0\}\cup\bigcup_i[2^{-2i-2},2^{-2i-1}]$ and $G=\omega\times G'$. Observe that F and G are regularly closed and that $\operatorname{int}F=\mathbb{M}\setminus G$ and $\operatorname{int}G=\mathbb{M}\setminus F$. Standard properties of the Čech–Stone compactification allow us to conclude that F^* and G^* are regularly closed as well and that $\operatorname{int}F^*=\mathbb{M}^*\setminus G^*$ and $\operatorname{int}G^*=\mathbb{M}^*\setminus F^*$. We see that it suffices to find a clopen subset C of ω^* such that $\operatorname{int}F^*\subseteq h[C]$ and $h[C]\cap\operatorname{int}G^*=\emptyset$.

We define, for $f\in{}^\omega\omega$

$$F_f=\bigcup_{n\in\omega}\bigcup_{i\le f(n)}\{n\}\times\bigl[2^{-2i-1}+2^{-f(n)},2^{-2i}-2^{-f(n)}\bigr]$$

and

$$G_f = \bigcup_{n\in\omega} \bigcup_{i\le f(n)} \{n\} \times \left[2^{-2i-2} + 2^{-f(n)}, 2^{-2i-1} - 2^{-f(n)}\right].$$

It is readily seen that $\operatorname{int} F^* = \bigcup_{f\in{}^\omega\omega} F_f^*$ and likewise for $\operatorname{int} G^*$.

Now let $\mathcal{I}_F$ denote the family of those subsets A of ω for which $h[A^*] \subseteq \operatorname{int} F^*$ and define $\mathcal{I}_G$ similarly.

We use [8, Theorem 8.6] to show that OCA implies that both $\mathcal{I}_F$ and $\mathcal{I}_G$ are $P_{\aleph_1}$-ideals, i.e., whenever $\mathcal{I}_F'$ is an $\aleph_1$-sized subfamily of $\mathcal{I}_F$ there is an A in $\mathcal{I}_F$ such that $B^* \subseteq A^*$ for all B in $\mathcal{I}_F'$ and similarly for $\mathcal{I}_G$. Indeed, for every $A \in \mathcal{I}_F$ there is an f such that $h[A^*] \subseteq F_f^*$ and, conversely, for every f there is an $A \in \mathcal{I}_F$ such that $F_f^* \subseteq h[A^*]$. Furthermore, OCA implies, by the result cited above, that for every $\aleph_1$-sized subfamily $\mathcal{F}$ of ${}^\omega\omega$ there is a $g \in {}^\omega\omega$ such that $f <^* g$ for every $f \in \mathcal{F}$.

The family $\mathcal{I} = \{K \cup L : K \in \mathcal{I}_F \text{ and } L \in \mathcal{I}_G\}$ is also a $P_{\aleph_1}$-ideal and we can choose for every $I \in \mathcal{I}$ a function $f_I : I \to \{0,1\}$ such that $f_I^{\leftarrow}(0) \in \mathcal{I}_F$ and $f_I^{\leftarrow}(1) \in \mathcal{I}_G$. This family is **coherent** in the sense that whenever $I, J \in \mathcal{I}$ the set $\{n \in I \cap J : f_I(n) \neq f_J(n)\}$ is finite. Now Theorem 8.7 from [8] applies and we can find one function $f\colon \omega \to \{0,1\}$ such that $f_I \subseteq^* f$ for all $I \in \mathcal{I}$.

One readily checks that $C = f^{\leftarrow}(0)^*$ is the required clopen subset of ω^*.

2. No simple mappings

In this section we show that a surjection of ω^* onto $\mathbb{D}^*$ cannot have a very simple structure. Later we shall show that OCA implies that such surjections **must** have such a simple structure, thus showing that they cannot exist under this assumption.

First of all we give a description of the Boolean algebra of clopen subsets of $\mathbb{D}$ that is easy to work with. We work in $\omega \times \omega$ and denote the n-th column $\{n\} \times \omega$ by C_n. The family

$$\mathcal{B} = \left\{X \subseteq \omega \times \omega : (\forall n \in \omega)(C_n \subseteq^* X \vee C_n \cap X =^* \emptyset)\right\}$$

is the Boolean algebra of clopen subsets of $\mathbb{D}$. We also consider the subfamily

$$\mathcal{B}^- = \left\{X \in \mathcal{B} : (\forall n \in \omega)(C_n \cap X =^* \emptyset)\right\}$$

of $\mathcal{B}$.

Now assume $S\colon \omega^* \to \mathbb{D}^*$ is a continuous surjection and take a map $\Sigma : \mathcal{B} \to \mathcal{P}(\omega)$ that represents S, i.e., for all $X \in \mathcal{B}$ we have $\Sigma(X)^* = S^{\leftarrow}[X^*]$. Note that if X is compact in $\mathbb{D}$ then $\Sigma(X)$ is finite.

The main result of this section is that Σ cannot be simple, where simple maps are defined as follows.

Definition 2.1: We call a map $F\colon \mathcal{B}^- \to \mathcal{P}(\omega)$ **simple** if there is a map f from $\omega \times \omega$ to $[\omega]^{<\omega}$ such that $F(X) =^* f[X]$ for all X, where $f[X]$ denotes the set $\bigcup_{x\in X} f(x)$.

THEOREM 2.2: *The map $\Sigma \restriction \mathcal{B}^-$ is not simple.*

Proof: We assume that there is a map $\sigma\colon \omega\times\omega \to [\omega]^{<\omega}$ such that $\sigma[X] =^* \Sigma(X)$ for all X; this implies that $\sigma[X]^* = S^{\leftarrow}[X^*]$ for all X, so the map $X \mapsto \sigma[X]$ also represents S. We may therefore as well assume that $\Sigma(X) = \sigma[X]$ for all X.

CLAIM 1: *We can assume that the values $\sigma(x)$ are pairwise disjoint.*

Let $\langle f_\alpha : \alpha < \mathfrak{b}\rangle$ be a sequence in ${}^\omega\omega$ that is strictly increasing and unbounded with respect to $<^*$; also each f_α is assumed to be strictly increasing.

For each α let $L_\alpha = \{(n,m) : m \le f_\alpha(n)\}$ and $A_\alpha = \sigma[L_\alpha]$. Next let

$$B_\alpha = \{i \in A_\alpha : (\exists x, y \in L_\alpha)(x \ne y \wedge i \in \sigma(x) \cap \sigma(y))\}.$$

Now if B_α were infinite then we could find different i_n in B_α and different x_n and y_n in L_α such that $i_n \in \sigma(x_n) \cap \sigma(y_n)$. But then $X = \{x_n : n \in \omega\}$ and $Y = \{y_n : n \in \omega\}$ would be disjoint yet $\sigma[X] \cap \sigma[Y]$ would be infinite.

We conclude that each B_α is finite and because $\mathfrak{b}$ is regular we can assume that all B_α are equal to the same set B. Fix n such that $[n,\omega) \times \omega \subseteq \bigcup_\alpha L_\alpha$ and note that on $[n,\omega) \times \omega$ we have $\sigma(x) \cap \sigma(y) \subseteq B$ whenever $x \ne y$. Replace $\sigma(x)$ by $\sigma(x) \setminus B$ and $\omega \times \omega$ by $[n,\infty) \times \omega$.

In a similar fashion we can prove the following claim.

CLAIM 2: *We can assume that the values $\sigma(x)$ are all nonempty.*

There are only finitely many n for which there is an m such that $\sigma(n,m) = \emptyset$. Otherwise we could find a noncompact $X \in \mathcal{B}^-$ for which $\Sigma(X) = \emptyset$. Drop these finitely many columns from $\omega \times \omega$.

For each n let $D_n = \sigma[C_n]$ and work inside $D = \bigcup_n D_n$. Also define, for $f \in {}^\omega\omega$, the sets $L_f = \{(n,m) : m \le f(n)\}$ and $M_f = \sigma[L_f]$.

Now observe the following: for each f and n the intersection $M_f \cap D_n$ is finite and if $X \subseteq D$ is such that $X \cap D_n =^* \emptyset$ for all n then $X \subseteq M_f$ for some f.

In $\mathbb{D}^*$ we consider the top line $T = (\omega \times \{\omega\})^*$ and its complement O. First we note that $O = \bigcup_f L_f^*$ and so

$$S^{\leftarrow}[O] = \bigcup_f S^{\leftarrow}[L_f^*] = \bigcup_f \sigma[L_f]^* = \bigcup_f M_f^*.$$

This means that $S[D_n^*] \subseteq T$ for all n, because D_n^* is disjoint from $\bigcup_f M_f^*$. Also, the boundary of the cozero set $\bigcup_n D_n^*$ is the boundary of $\bigcup_f M_f^*$; by continuity this boundary is mapped onto the boundary of O, which is T.

This argument works for every infinite subset A of ω: the boundary of $\bigcup_{n\in A} D_n^*$ is mapped exactly onto the set $T_A = \bigl(A \times \{\omega\}\bigr)^*$ and so T_A is contained in the closure of $S[\bigcup_{n\in A} D_n^*]$ and $S[D_n^*] \subseteq T_A$ for all but finitely many $n \in A$.

From the fact that nonempty G_δ-sets in ω^* have nonempty interior one readily deduces that no countable family of nowhere dense subsets of ω^* has a dense union. We conclude that there is an n_0 such that $\operatorname{int}_T S[D_{n_0}^*]$ is nonempty. Choose an infinite subset A_0 of (n_0, ω) such that $T_{A_0} \subseteq S[D_{n_0}^*]$.

Continue this process: once n_i and A_i are found one finds $n_{i+1} \in A_i$ such that $S[D_{n_{i+1}}^*]$ has nonempty interior and is contained in T_{A_i}, next choose an infinite subset A_{i+1} of $A_i \cap (n_{i+1}, \omega)$ such that $T_{A_{i+1}} \subseteq S[D_{n_{i+1}}^*]$.

Finally then let $A = \{n_{2i} : i \in \omega\}$ and $B = \{n_{2i+1} : i \in \omega\}$. Note that $T_A \subseteq \bigcap_{n\in B} S[D_n^*]$ but also that $S[D_n^*] \cap T_A = \emptyset$ for all but finitely many $n \in B$. This contradiction completes the proof of Theorem 2.2. ∎

3. All maps must be simple

In this section we finish the proof of our main result by showing that, under OCA, every map Σ that represents S must be simple. For this we must localize the notion of simplicity.

Definition 3.1: Let $B \subseteq \omega \times \omega$ and let $F : \mathcal{B}^- \to \mathcal{P}(\omega)$ be any map. We call F **simple on** B if there is $\sigma \colon B \to [\omega]^{<\omega}$ such that $F(X) =^* \sigma[X]$ for all subsets X of B that are in $\mathcal{B}^-$.

The next proposition tells us that simple is the same as locally simple.

PROPOSITION 3.2 (OCA): *A map $F \colon \mathcal{B}^- \to \mathcal{P}(\omega)$ is simple iff it is simple on L_h for every $h \in {}^\omega\omega$.*

Proof: Assume there is for each h a witness σ_h to the simplicity of F on L_h. It should be clear that for any two functions h and k the maps differ on $L_h \cap L_k$ in only finitely many points. As at the end of Section 1 we apply Theorem 8.7 from [8] to find one map σ on $\omega \times \omega$ such that for all h we have $\sigma \restriction L_h =^* \sigma_h$. Clearly σ witnesses that F is simple on $\mathcal{B}^-$. ∎

An obvious consequence of Theorem 2.2 is that our map Σ is not simple on any set of the form $A \times \omega$. Therefore we can find, by Proposition 3.2, for every infinite subset A of ω a function f_A such that Σ is not simple on $L_{f_A} \cap A \times \omega$.

We shall obtain a contradiction by showing that Σ must be trivial on one of the sets $L_{f_A} \cap (A \times \omega)$. We follow the strategy laid out in Veličković papers [9] and [10]. In what follows we shall assume that the reader has these two papers on hand. In the proof we consider a power set $\mathcal{P}(X)$ as a topological space by identifying it with the Cantor cube 2^X. Terms such as 'continuity' and 'Borel measurable' will be used with respect to this topology and its corresponding family of Borel sets.

First fix a bijection c from ω onto the binary tree $2^{<\omega}$ and choose an almost disjoint family $\mathcal{A} = \{A_\alpha : \alpha < \omega_1\}$ of subsets of ω such that each image $c[A_\alpha]$ is a branch through $2^{<\omega}$ — in [10] such a family is called **neat**. Next fix, as in the penultimate paragraph of Section 1, one function f such that $f_{A_\alpha} <^* f$ for all α. We abbreviate $L_f \cap (A_\alpha \times \omega)$ by L_α.

To apply OCA we need a separable metric space; we take

$$X = \{\langle a, b\rangle : (\exists \alpha < \omega_1)(b \subseteq a \subseteq L_\alpha)\},$$

topologized by identifying $\langle a, b\rangle$ with $\langle a, b, \Sigma(a), \Sigma(b)\rangle$ — that is, X is identified with a subset of $\mathcal{P}(\omega)^4$. We define a partition $[X]^2 = K_0 \cup K_1$ by: $\{\langle a, b\rangle, \langle c, d\rangle\} \in K_0$ iff (1) a and c are in different L_α's; (2) $a \cap d = c \cap b$, and (3) $\Sigma(a) \cap \Sigma(d) \neq \Sigma(c) \cap \Sigma(b)$.

One uses neatness of the family $\mathcal{A}$ to show that K_0 is open. The proof of [10, Lemma 2.2] now applies: there is no uncountable K_0-homogeneous set, so X is the union of countably many K_1-homogeneous sets. This then implies that for all but countably many α the restriction of Σ to L_α can be covered by countably many Borel maps. We may now apply Proposition 5.2 from the Appendix to see that Σ is simple on L_α and hence on $L_{f_{A_\alpha}} \cap (A_\alpha \times \omega)$ for those α's. This proposition generalizes Theorem 1.2 from [10]; the proofs are almost identical.

4. Questions

The proof of the second reduction raises the obvious question whether one really needs OCA to show that ω^* maps onto $\mathbb{D}^*$ if it maps onto $\mathbb{H}^*$ or, equivalently, onto $\mathbb{M}^*$.

QUESTION 4.1: *If ω^* maps onto $\mathbb{H}^*$ then does it map onto $\mathbb{D}^*$?*

It would be very intriguing indeed if it were consistent that ω^* does map onto $\mathbb{H}^*$ but not onto $\mathbb{D}^*$. In that case ω^* would not map onto $(\omega \times 2^\omega)^*$ either and we would have a map from ω^* onto $\mathbb{M}^*$ that would not lift to a map onto $(\omega \times 2^\omega)^*$ via the obvious quotient mapping (i.e., the map induced by the familiar surjection of the Cantor set onto the unit interval).

We proved that ω^* maps onto $\mathbb{M}^*$ if it maps onto $\mathbb{H}^*$ mainly for convenience. What we could not decide however was the following:

QUESTION 4.2: *Does every map from ω^* onto $\mathbb{H}^*$ lift to a map onto $\mathbb{M}^*$?*

The lifting we are seeking is via the Čech–Stone extension of the 2-to-1-surjection from $\mathbb{M}$ onto $\mathbb{H}$ defined by $q(n,x) = n + x$. The same argument as in Section 1 will show that the answer is yes under OCA: one gets a clopen set C that maps onto $\left(\bigcup_n [2n, 2n+1]\right)^*$ and whose complement maps onto $\left(\bigcup_n [2n+1, 2n+2]\right)^*$. Of course, we have just proved that OCA implies that there is no continuous map from ω^* onto $\mathbb{H}^*$ so this seems vacuous but in the reduction we did not use the full force of OCA.

Regarding $(\omega \times 2^\omega)^*$ we have the following two questions:

QUESTION 4.3: *If ω^* maps onto $\mathbb{H}^*$ then does it map onto $(\omega \times 2^\omega)^*$?*

QUESTION 4.4: *Does every map from ω^* onto $\mathbb{M}^*$ lift to a map onto $(\omega \times 2^\omega)^*$.*

This lifting should go via the usual 2-to-1 surjection of the Cantor set onto the unit interval.

5. Appendix: Modifying Veličković's proof

Many mathematicians have noticed that more can be deduced from Veličković's proof than is stated in [10]. However, to our knowledge, the specific result that we need is not available in the literature. We therefore include this appendix for the convenience of the interested reader. We also note that this and many other modifications of [10] were obtained by Farah in [2].

We remind the reader that we assume she has [9] and [10] on hand. We also recall that we have a surjection $S : \omega^* \to \mathbb{D}^*$ and a map $\Sigma : \mathcal{B} \to \mathcal{P}(\omega)$ that represents S, in the sense that $\Sigma[X]^* = S^{\leftarrow}[X^*]$ for all elements of $\mathcal{B}$. Our goal, as announced at the end of Section 3, is to prove Proposition 5.2 below.

PROPOSITION 5.1 (Compare [9, Lemma 2]): *If $B \in \mathcal{B}^-$ and if Σ is continuous on $\mathcal{P}(B)$ or even on a dense G_δ-subset of $\mathcal{P}(B)$ then it is simple on B.*

The proof is identical to the one given in [9] except that there is no need to show that for almost all $x \in B$ the set $\sigma(x)$ consists of exactly one point.

PROPOSITION 5.2 (Compare [10, Theorem 1.2]): *If $B \in \mathcal{B}^-$ and if $\Sigma \restriction \mathcal{P}(B)$ can be covered by countable many Borel measurable maps then it is simple on B.*

The proof in [10] will work but with one important exception. To deal with this exception we let $\mathcal{I}$ denote the ideal of subsets of B on which Σ is simple.

At one point in the proof of the following lemma Veličković explicitly uses the fact that his map induces an automorphism of the Boolean algebra $\mathcal{P}(\omega)/\operatorname{fin}$; we show how to avoid this.

LEMMA 5.3 (Compare [10, Lemma 1.3]): *The ideal $\mathcal{I}$ is not a non-principal maximal ideal.*

Proof: As in the proof of Lemma 1.3 in [10] one can find a subset A of B such that Σ is not simple on A and such that $\Sigma \restriction \mathcal{P}(A)$ is covered by countably many **continuous** maps. One then lets $\mathcal{T}$ denote the family of subsets of A that are in $\mathcal{I}$; we fix for every T in $\mathcal{T}$ a map $\sigma_T\colon T \to [\omega]^{<\omega}$ such that $\Sigma(X) =^* \sigma_T[X]$ for all $X \subseteq T$. As in the proof of Theorem 2.2 we may assume that, for every T, the values $\sigma_T(x)$ are pairwise disjoint.

Still following [10] we find continuous maps $H_n\colon \mathcal{P}(A) \to \mathcal{P}(\omega)$ such that for every $T \in \mathcal{T}$ there is n for which $H_n(X) = \sigma_T[X]$ for all subsets X of T. We let $\mathcal{T}_n$ denote the set of those $T \in \mathcal{T}$ for which one can choose H_n.

As in [10] the assumption that some $\mathcal{T}_n$ is cofinal in $\langle \mathcal{T}, \subseteq^* \rangle$ leads to a contradiction. Therefore, we can partition A into sets T_n from $\mathcal{T}$ such that no $T \in \mathcal{T}$ almost contains all the T_n. By our tacit assumption that $\mathcal{I}$ is a non-principal maximal ideal we know that $\mathcal{U}$, the family of those subsets of A that are almost disjoint from all T_n, is a subfamily of $\mathcal{T}$. Moreover, this family $\mathcal{U}$ is σ-directed, so there is an n such that $\mathcal{U}_n = \mathcal{U} \cap \mathcal{T}_n$ is cofinal in $\langle \mathcal{U}, \subseteq^* \rangle$. Let $\sigma = \bigcup_{U \in \mathcal{U}_n} \sigma_U$; using H_n it follows that σ determines Σ on all elements of $\mathcal{U}$.

Just as in Claim 1 of Theorem 2.2 one proves that there is an n_0 such that $\sigma(x) \cap \sigma(y) = \emptyset$ whenever x and y are distinct elements of $\bigcup_{n \geq n_0} T_n$.

CLAIM 1: *Let T be an element of $\mathcal{T}$. Then there is an n_T such that $\sigma \restriction T' = \sigma_T \restriction T'$, where $T' = T \cap \bigcup_{n \geq n_T} T_n$.*

Indeed, assume that for infinitely many n there is $x_n \in T \cap T_n$ such that $\sigma(x_n) \neq \sigma_T(x_n)$. The set U of those x_n belongs to $\mathcal{U}$ and, if necessary, we can thin out U so as to get the unions $\sigma(x_n) \cup \sigma_T(x_n)$ pairwise disjoint. This gives a contradiction, because now $\sigma[U] \neq^* \sigma_T[U]$, whereas also $\sigma[U] =^* \Sigma(U) =^* \sigma_T[U]$.

Using this claim and the fact that $\mathcal{T}$ is a non-principal maximal ideal we find an $n_1 \geq n_0$ such that σ induces Σ on all T_n with $n \geq n_1$: if no such n_1 can be found we find infinitely many 'bad' T_n, say $\{T_n : n \in P\}$. Find an infinite subset P' of P such that $\bigcup_{n \in P'} T_n \in \mathcal{T}$; then all but finitely many of these T_n are 'good' anyway.

This now gives us our final contradiction: let $T \in \mathcal{T}$; write T as the union of

T' and the sets $T \cap T_n$ with $n < n_T$:

$$T = \bigcup_{n<n_1} (T \cap T_n) \cup \bigcup_{n_1 \le n < n_T} (T \cap T_n) \cup T'.$$

Apply Σ and use the facts established above to find:

$$\begin{aligned}
\Sigma(T) &=^* \bigcup_{n<n_1} \Sigma(T \cap T_n) \cup \bigcup_{n_1 \le n < n_T} \Sigma(T \cap T_n) \cup \Sigma(T') \\
&=^* \bigcup_{n<n_1} \sigma_{T_n}[T \cap T_n] \cup \bigcup_{n_1 \le n < n_T} \sigma[T \cap T_n] \cup \sigma[T'] \\
&=^* \bigcup_{n<n_1} \sigma_{T_n}[T \cap T_n] \cup \sigma\left[T \setminus \bigcup_{n<n_1} T_n\right].
\end{aligned}$$

Using the fact that $\mathcal{T}$ is a non-principal maximal ideal once more we find, using complements, that this formula holds for all subsets of A. This contradiction completes the proof. ∎

As in [10] one now uses Proposion 5.1 to finish the proof of Proposition 5.2.

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